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THE STABILITY OF CERTAIN RETARDED SYSTEMS WITH VARIABLE COEFFICIENTS†

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A procedure for studying some retarded systems with time-dependent parameters is proposed and the stability conditions for such systems are established.

Lyapunov's second method, which was developed for systems with finite delay [1], is one of the basic techniques for studying the stability of retarded systems. Many papers, for example, [2–9] and references therein, have been devoted to further modifications of this method and its applications. In a number of these papers coefficient conditions for stability have been obtained for specific retarded systems by constructing suitable Lyapunov functionals. However, the difficulties which had to be overcome made various authors compare the construction of these functionals to an art [2].

Along with this, if the delay is set to zero in the functionals constructed, one obtains the Lyapunov function for the corresponding ordinary differential equation. In other words, every Lyapunov functional for a retarded system is generated by the Lyapunov function of an ordinary differential equation constructed in the appropriate way. This observation was used in [10], where a formal procedure for constructing Lyapunov functionals V for stationary systems with arbitrary (concentrated or distributed) delay was put forward. The procedure consists of the following steps:

1. transform the right-hand side of the equation in question so as to represent it as the sum of two terms, the first of which depends only on the current state of the system;
2. discard all but the first term of the transformed equation, resulting in an auxiliary ordinary differential equation, whose trivial solution is assumed to be asymptotically stable;
3. for the auxiliary system, determine the Lyapunov function V (i.e. a positive definite function, whose derivative is negative definite by virtue of the auxiliary system), the existence of which follows from the assumptions of step (2);
4. replace the arguments of V by the functionals depending on the transformation used in (1); as a result, one obtains the main component V_1 of V; by adding to V_1 a component V_2 obtained in the standard way, we obtain the desired functional V of the form $V = V_1 + V_2$, which satisfies the conditions of the theorem on asymptotic stability.

The transformations in (1) and the choice of the Lyapunov function in (3) can, in general, be realized in many ways. This lack of uniqueness can be explained to construct different functionals V and, consequently, to obtain different stability conditions.

We remark that the functionals constructed earlier in [4, 5] are within the framework of the formal procedure presented for studying the stability of retarded systems.

The purpose of the present paper is to obtain stability conditions and to demonstrate that the procedure described is also applicable to a number of systems with variable coefficients of the form

$$\dot{x}(t) = a(t, x_t), \quad t \geqslant 0, \quad x \in R^n, \quad a(t, 0) \equiv 0 \tag{0.1}$$

Here $x_t = x(t + \theta)$, $\theta \leqslant 0$ and $a(t, \cdot)$: $[0, \infty] \times C(-\infty, 0] \to R^n$.

The initial conditions for (1.1) are defined by

$$x_0(\theta) = \varphi(\theta), \quad \varphi \in C(-\infty, 0] \tag{0.2}$$

Below we consider the stability of the trivial solution of a problem of the form (0.1), (0.2), which is said to be:

1. stable if for any $\varepsilon > 0$ a quantity $\delta(\varepsilon) > 0$ exists such that $x(t)| < \varepsilon$ for $t \geqslant 0$ if $\|\varphi\| < \delta(\varepsilon)$;
2. asymptotically stable if it is stable and $|x(t)| \to 0$ as $t \to \infty$ for all initial conditions φ belonging to some attraction domain of the trivial solution.

We will first consider in detail a scalar linear equation with discrete delay as an illustration of the procedure. As regards other classes of equations, we shall confine ourselves to presenting the results of the various stages of the procedure, to the form of the resulting functionals, and to the stability conditions. For convenience, we shall consider only simple characteristic cases. Nevertheless, the results presented show clearly what changes need to be made in order to cover a more general situation.

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1. SCALAR EQUATIONS

Consider the stability of the scalar equation

$$\dot{x}(t) = -b(t)x(t-h), \quad t \geq 0 \tag{1.1}$$

where $h \geq 0$ is a given constant and $b(t)$ is a given bounded continuous function. We transform the right-hand side of (1.1), separating the term which depends on $x(t)$

$$b(t)x(t-h) = b(t+h)x(t) - \frac{d}{dt}\int_{t-h}^{t} b(s+h)x(s)ds \tag{1.2}$$

We substitute (1.2) into (1.1) and transfer all the terms containing the t-derivative to the left-hand side. Then the transformed equation can be represented in the form

$$\dot{z}(t) = -b(t+h)x(t), \quad t \geq 0, \quad z(t) = x(t) - \int_{t-h}^{t} b(s+h)x(s)ds \tag{1.3}$$

Discarding the term that contains retarded values of the solution in (1.3), we obtain the auxiliary ordinary equation

$$\dot{y}(t) = -b(t+h)y(t), \quad t \geq 0 \tag{1.4}$$

We assume that

$$\inf_t b(t) > 0, \quad t > 0 \tag{1.5}$$

Then (1.4) is uniformly (with respect to the initial instant) asymptotically stable. As the Lyapunov function $v(t, y)$ for (1.4) we take $v = y^2$. Now, we form the component V_1 of V. The functional V_1 is always obtained by replacing y by z in $v(t, y)$, z being the functional under the derivative sign on the left-hand side of the transformed equation.

In the case under consideration we find, on the basis of (1.3), that

$$V_1(t, x_t) = \left[x(t) - \int_{t-h}^{t} b(s+h)x(s)ds\right]^2 \tag{1.6}$$

Now we choose V_2 in such a way that the complete derivative $\dot{V}$ of $V = V_1 + V_2$ is negative definite by (1.1).

Taking (1.2) into account, we conclude that the derivative $\dot{V}_1$ of (1.6) is, by (1.1), equal to

$$\dot{V}_1(t) = -2z(t)b(t+h)x(t) = -2b(t+h)x^2(t) + 2b(t+h)x(t)\int_{t-h}^{t} b(x+h)x(s)ds$$

Besides

$$2x(t)\int_{t-h}^{t} b(s+h)x(s)ds \leq x^2(t)\int_{t-h}^{t} b(s+h)ds + \int_{t-h}^{t} b(s+h)x^2(s)ds \tag{1.7}$$

Therefore, the functional

$$V_2 = \int_{t-h}^{t} b(s+2h)ds \int_{s}^{t} b(s_1+h)x^2(s_1)ds_1 \tag{1.8}$$

must be taken as V_2. By (1.6)–(1.8), we have

$$\dot{V} \leq -x^2(t)b(t+h)[2 - \int_{t}^{t+2h} b(s)ds] \tag{1.9}$$

for the derivative $\dot{V}$ of $V = V_1 + V_2$. From (1.5) and (1.9) it follows that $\dot{V}$ is positive definite under the condition

$$\sup_t \int_t^{t+h} b(s)ds < 1, \quad t \geqslant 0 \tag{1.10}$$

Now, a standard argument (see, for example, [4]) shows that (1.4) is asymptotically stable for any continuous function $b(t)$ that satisfies (1.5) and (1.10).

Using other transformations of the right-hand side of (1.1), we can obtain other stability conditions. For example, consider the following equation (valid for $t \geqslant h$ under the assumption that $b(s)$ is a continuously differentiable function)

$$b(t)x(t-h) = b(t+h)x(t) - \int_{t-h}^{t} [\dot{b}(s+h)x(s) - b(s+h)b(s)x(s-h)]ds, \quad t \geqslant h \tag{1.11}$$

By (1.1) and (1.11), we conclude that the auxiliary system has the previous form (1.4). This means that, when (1.5) holds, $v(t, y) = y^2$, i.e. $V_1 = x^2(t)$, since in this case $z(t) = x(t)$ by (1.11). Computing $\dot{V}_1$ from (1.1) and taking (1.11) into account, we have

$$\dot{V}_1(t) = -2b(t+h)x^2(t) + 2x(t)\int_{t-h}^{t} [\dot{b}(s+h)x(s) - b(s+h)b(s)x(s-h)]ds \leqslant$$

$$\leqslant x^2(t)\left[-2b(t+h) + \int_{t-h}^{t} [|\dot{b}(s+h)| + b(s+h)b(s)]ds + \right.$$

$$\left. + \int_{t-h}^{t} [|\dot{b}(s+h)|x^2(x) + b(s+h)b(s)x^2(s-h)]ds \right. \tag{1.12}$$

It follows that to ensure that $\dot{V}$, where $V = V_1 + V_2$, is negative definite, it suffices to take V_2 to be

$$V_2 = \int_{t-h}^{t} d\tau \int_{\tau}^{t} [|\dot{b}(s+h)|x^2(s) + b(s+h)b(s)x^2(s-h)]ds + h\int_{t-h}^{t} b(s+2h)b(s+h)x^2(s)ds \tag{1.13}$$

Indeed, (1.12) and (1.13) imply that

$$\dot{V}(t) \leqslant \gamma(t)x^2(t),$$

$$\gamma(t) = [-2b(t+h) + h|\dot{b}(t+h)| + hb(t+2h)b(t+h) + \int_{t-h}^{t} (|\dot{b}(s+h) + b(s+h)b(s))ds]$$

It follows that (1.1) is asymptotically stable for any continuously differentiable function $b(t)$ satisfying (1.5) such that

$$\sup_t \gamma(t) < 0, \quad t \geqslant 0$$

Remark. As has been mentioned above, (1.1) is considered purely for simplicity. Minor modifications enable stability conditions for more general equations to be obtained using the same procedure. As an example, we consider the following scalar equation with variable delay

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i(t)x(t-\tau_i(t)), \quad t \geqslant 0 \tag{1.14}$$

The transformed system has the form

$$\dot{z}(t) = -b(t)x(t), \quad b(t) = \sum_{i=1}^{m} a_i(q_i(t))\dot{q}_i(t)$$

$$z(t) = x(t) - \sum_{i=1}^{m} \int_{t}^{q_i(t)} a_i(s)x(s-\tau_i(s))ds$$

where $q_i(t)$ is the inverse function to $t - \tau_i(t)$. This means that the auxiliary system can be described by the equation $\dot{y}(t) = -b(t)y(t)$ with Lyapunov's function $v = y^2$. Consequently, $V = V_1 + V_2$, where

$$V_1 = z^2(t), \quad V_2 = \sum_{i=1}^{m} \int_{t}^{q_i(t)} b(s)ds \int_{s}^{q_i(t)} |a_i(s_1)|x^2(s_1 - \tau_i(s))ds_1$$

Using V, it can be established that (1.14) is asymptotically stable if $a_i(t)$ are bounded and continuous, $\tau_i(t)$ are continuously differentiable, and

$$\sup_t \dot{\tau}_i(t) < 1, \quad \sup_t \sum_{i=1}^{m} \int_t^{q_i(t)} |a_i(s)ds < 1, \quad t \geq 0$$

$$\sup_t [-2b(t) + \sum_{i=1}^{m} \int_t^{q_i(t)} (|a_i(s)b(t)| + |b_i(s)a_i(q_i(t))\dot{q}_i(t)|ds] < 0$$

2. SECOND-ORDER EQUATIONS

We shall establish stability conditions for the system

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -a(t)x_2(t) - b(t)x_1(t-\tau(t)), \quad t \geq 0 \tag{2.1}$$

The transformed system has the form

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -a(t)x_2(t) - b(t)x_1(t) + b(t) \int_{t-\tau(t)}^{t} x_2(s)ds \tag{2.2}$$

It follows that the auxiliary system can be described by

$$\dot{y}_1(t) = y_2(t), \quad \dot{y}_2(t) = -a(t)y_2(t) - b(t)y_1(t)$$

For this system, we take v to be $v = by_1^2 + y_2^2$. Therefore $V_1 = bx_1^2 + x_2^2$.

This means that, by (2.1) $\dot{V}_1$ satisfies the estimate

$$V_1 \leq \dot{b}(t)x_1^2(t) - 2a(t)x_2^2(t) + b(t)\tau(t)x_2^2(t) + b(t) \int_{t-\tau(t)}^{t} x_2^2(s)ds$$

It follows that

$$V_2 = \int_t^{q(t)} b(s)ds \int_{s-\tau(s)}^{t} x_2^2(s_1)ds_1 \tag{2.3}$$

where $q(t)$ is the inverse function to $t - \tau(t)$. Taking (2.3) for $V = V_1 + V_2$, we have

$$\dot{V} \leq \dot{b}(t)x_1^2(t) + \gamma_1 x_1^2, \quad \gamma_1 = [-2a(t) + b(t)\tau(t) + \int_t^{q(t)} b(s)ds]$$

Therefore, using the transformed system (2.2) we can conclude that (2.1) is stable if $b(t)$, $\tau(t) \geq 0$ are continuously differentiable functions, $a(t)$ is continuous, and

$$\inf_t b(t) > 0, \quad \dot{b}(t) \leq 0, \quad \dot{\tau}(t) < 1, \quad \gamma_1(t) \leq 0, \quad t \geq 0 \tag{2.4}$$

We now take the transformed system in the form

$$\dot{x}_1(t) = x_2(t), \quad \dot{z}_2(t) = -\alpha(t)x_1(t) - a(t)x_2(t)$$

$$z(t) = x_2(t) - \int_t^{q(t)} b(s)x_1(s-\tau(s))ds, \quad \alpha(t) = b(q(t))\dot{q}(t) \tag{2.5}$$

Then the auxiliary system can be described by

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -\alpha(t)x_1(t) - a(t)x_2(t)$$

We take v in the form

$$v = 2\alpha y_1^2 + (\alpha y_1 + y_2)^2 + y_2^2$$

Thus, taking (2.5) into account

$$V_1 = 2\alpha x_1^2(t) + z^2(t) + [z(t) + ax_1(t)]^2 \tag{2.6}$$

By (2.1), we have

$$\tfrac{1}{2}\dot{V}_1 = [\dot{\alpha}(t) + a(t)(\dot{a}(t) - \alpha(t))]x_1^2(t) - a(t)x_2^2(t) + \dot{a}(t)x_1(t)x_2(t) +$$
$$+[(2\alpha(t) - \dot{a}(t))x_1(t) + a(t)x_2(t)]\int_t^{q(t)} b(s)x_1(s - \tau(s))ds \tag{2.7}$$

for $\dot{V}_1$. It follows that

$$V_2 = \int_t^{q(t)} [|2\alpha(s) - \dot{a}(s)| + a(s)]ds \int_t^{q(t)} b(s_1)x_1(s_1 - \tau(s_1))ds_1$$

Hence, (2.6) and (2.7) imply that

$$\dot{V} \leqslant \gamma_3(t)x_1^2 + \gamma_2(t)x_2^2(t) \tag{2.8}$$

Here

$$\gamma_3 = 2(\dot{\alpha} + a(\dot{a} - \alpha)) + |2\alpha - \dot{a}|\int_t^{q(t)} b(s)ds + \alpha\int_t^{q(t)} (|2\alpha(s) - \dot{a}(s)| + |\dot{a}| + a(s)]ds,$$

$$\gamma_2 = -2a + a\int_t^{q(t)} b(s)ds + |\dot{a}|$$

The relationships (2.6)–(2.8) demonstrate that (2.1) is asymptotically stable if $\dot{a}, \dot{b}, \dot{\tau}, \ddot{\tau}$ are continuous functions and

$$a(t) \geqslant 0, \quad \sup_t \gamma_i(t) < 0, \quad i = 2,3, \quad \sup_t \dot{\tau}(t) < 1, \quad \inf_t \alpha(t) > 0, \quad t \geqslant 0$$

Note that the two groups of stability conditions for (2.1) established in this section depend not only on the coefficients, but also on their derivatives.

Remark. The above procedure for constructing the functionals can be applied not just to linear, but also to some non-linear equations. For example, consider the system

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -a(t)x_2(t) - b(t)F(x_1(t - \tau(t))) \tag{2.9}$$

The function $F(x)$ is continuously differentiable, $xF(x) > 0$ for $x \neq 0$, and $f(x) = \partial F(x)/\partial x$ is bounded: $|f(x)| < 1$. The remaining parameters in (2.9) are the same as in (2.1). We shall construct $V = V_1 + V_2$ for (2.9). The transformed system has the form

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -a(t)x_2(t) - b(t)F(x_1(t)) + b(t)\int_{t-\tau(t)}^{t} f(x_1(s))x_2(s)ds$$

This means that the auxiliary system of equations has the form

$$\dot{y} = y_2, \quad \dot{y}_2(t) = -ay_2 - bF(y_1) \tag{2.10}$$

We take v for (2.10) to be [11]

$$v(t, y_1, y_2) = y_2^2 + 2b(t)\int_0^{y_1} F(s)ds$$

Then $V_1 = v(t, x_1, x_2)$. The functional defined by (2.3) should therefore be taken as V_2. As a result, we find that $\dot{V}$ satisfies the estimate

$$\dot{V} \leqslant 2\dot{b}(t)\int_0^{x_1(t)} F(s)ds + \gamma_1 x_2^2(t)$$

by (2.9). It follows that (2.9) is stable under the assumptions made above if conditions (2.4) are satisfied.

3. SYSTEMS WITH DISTRIBUTED DELAY

We shall apply the procedure described to the system of equations

$$\dot{x}(t)=\int_0^h K(t,s)x(t-s)ds,\quad t\geqslant 0,\quad x\in R^n \tag{3.1}$$

Here K is an $n\times n$-matrix with continuous elements. When $t\geqslant h$, the transformed system has the form

$$\dot{x}(t)=A(t)x(t)+\int_0^h K(t,s)ds\int_{t-s}^{t}d\tau\int_0^h K(\tau,s_1)x(t-s_1)ds_1,\quad A(t)=\int_0^h K(t,s)ds,\ t\geqslant 0 \tag{3.2}$$

The auxiliary system can be described by the equation

$$\dot{y}(t)=A(t)y(t),\ t\geqslant 0 \tag{3.3}$$

We assume that $A(t)$ is a continuously differentiable matrix, $|A(t)|\leqslant C_1$, and $\operatorname{Re}\lambda(A(t))\leqslant -C_2<0$ for all $t\geqslant 0$. Here and henceforth C_i are positive constants, $|A|$ is the matrix norm of A generated by the Euclidean vector norm in R^n, and, finally, $\lambda(A)$ are the eigenvalues of A, Re λ being their positive parts. Under these assumptions, a unique positive definite matrix $P(t)$ exists which is a solution of the equation $A'(t)P(t)+P(t)A(t)=-I$ for every $t\geqslant 0$, where the prime denotes transposition and I is the identity matrix. The matrix $P(t)$ satisfies the conditions [12–14]

$$\begin{gathered}|\dot{P}|\leqslant C_3|\dot{A}|,\ |P|\leqslant C_4,\ \ \upsilon(t,y)=y'P(t)y\\ C_3\leqslant 2\sqrt{n}|[A(t)\oplus A(t)]^{-1}|^2,\ \ C_4\leqslant\sqrt{n}|A'(t)\oplus A'(t)]^{-1}|\\ \dot{v}(t,y)=d/dt[y'P(t)y]=-|y|^2+y'\dot{P}y\end{gathered} \tag{3.4}$$

Here $\dot{v}$ is the total derivative of v by virtue of (3.3) and $\oplus$ denotes the Kronecker sum of the matrices A. By (3.2) and (3.3), we have

$$V_1=x'(t)P(t)x(t) \tag{3.5}$$

By virtue of (3.2), computing $\dot{V}_1$ we conclude using (3.3) that V_2 should be chosen in the form

$$\begin{gathered}V_2=C_4\int_0^h ds\int_{t-s}^{t}d\tau_1\int_{\tau_1}^{t}|K(\tau_1+s,s)|d\tau_2\int_0^h|K(\tau_2,s_1)||x(\tau_2-s_1)|^2ds_1+\\ +C_4\int_0^h ds\int_{t-s}^{t}[\int_0^h ds_1\int_{\tau+s-s_1}^{\tau+s}|K(\tau_1+s_1,s_1)|d\tau_1]|K(\tau+s,s)||x(\tau)|^2d\tau\end{gathered}$$

Then for $V=V_1+V_2$ we have

$$\begin{gathered}\dot{V}\leqslant-\gamma_4|x|^2,\ \ \gamma_4=1-|\dot{P}|-C_4Q\\ Q=\int_0^h ds\int_0^h ds_1\int_{t+s-s_1}^{t+s}|K(\tau_1+s_1,s_1)||K(t+s,s)|d\tau_1+\int_0^h|K(t,s)|ds\int_{t-s}^{t}d\tau\int_0^h|K(\tau,s_1)|ds_1\end{gathered} \tag{3.6}$$

It follows that (3.1) is asymptotically stable if $A(t)$ satisfies the above conditions for $\inf_t\gamma_4(t)>0$ for $t\geqslant 0$.

The condition $\gamma_4>0$ means that the moments of the kernel K and the rate of change of $A(t)$ must be small enough.

The stability conditions can be simplified for $N=1$. In this case the transformed and auxiliary systems will retain the previous form (3.2) and (3.3). We set $v=y^2$. Then $V_1=x^2(t)$. This means that V_2 is given by (3.6) with $C_4=1$. It follows that (3.1) is asymptotically stable for $n=1$ if

$$\sup_t[2A(t)+Q(t)]<0,\quad t\geqslant 0$$

As before, using a transformation formula other than (3.2), one can obtain other stability conditions. Let us state some them, confining ourselves to the case $n=1$ for simplicity. We write the transformed system as follows:

$$\dot{z}(t) = A_1(t)x(t), \quad A_1(t) = \int_0^h K(t+s,s)ds, \quad z(t) = x(t) + \int_0^h ds \int_{t-s}^t K(\tau+s,s)x(\tau)d\tau \tag{3.7}$$

It follows from (3.7) that the auxiliary system has the form (3.3) with A replaced by A_1. We assume that $A_1 \leq 0$. Then $v = y^2$, i.e. $V_1 = z^2(t)$.

Computing $\dot{V}_1$ by (3.1), we conclude that

$$V_2 = \int_0^h ds \int_{t-s}^t |A(s_1+s)|ds_1 \int_{s_1}^h |K(\tau+s,s)|x^2(\tau)d\tau \tag{3.8}$$

It follows that the asymptotic stability conditions for (3.1) have the form

$$\sup_t [2A_1(t) + |A_1(t)| \int_0^h ds \int_{t-s}^t |K(\tau+s,s)|d\tau + \int_0^h |K(t+s,s)|ds \int_{t-s}^t |A_1(s_1+s)|ds_1] < 0, \quad t \geq 0$$

$$\sup_t \int_0^h ds \int_{t-s}^t |K(\tau+s,s)|d\tau < 1 \tag{3.9}$$

Remarks. 1 Minor modifications of the original assumptions about $K(t, s)$ concerned with the existence and finiteness of improper integrals also make it possible to cover the case of infinite delay (i.e. the case $h = \infty$). For example, the asymptotic stability conditions for (3.1) for $h = \infty$ and $n = 1$ will retain the previous form (3.9) with h replaced by ∞ if

$$\sup_t \int_0^\infty ds \left(\int_{t-s}^\infty \left(\int_0^\infty |K(s_1+s+s_2,s_2)|ds_2 \right) ds_1 \int_{s_1}^t |K(\tau+s,s)|d\tau < \infty \right.$$

For the scalar equation

$$\dot{x}(t) = -a(t)x(t) + \int_0^\infty K(t,s)x(t-s)ds, \quad t \geq 0 \tag{3.10}$$

where $a(t)$ and $K(t, s)$ are continuous functions, the stability conditions have the form

$$\int_0^\infty ds \int_{t-s}^t |K(\tau+s,s)|d\tau < \infty \tag{3.11}$$

$$\sup_t [-2a(t) + \int_0^\infty |K(t,s)| + |K(t+s,s)|]ds < 0, \quad t \geq 0$$

Indeed, since the right-hand side of (3.10) already contains a term determined by the current state of the system, the auxiliary system has the form

$$\dot{y} = -ay, \text{ i.e. } v = y^2, \ V_1 = x^2(t)$$

Therefore

$$V_2 = \int_0^\infty ds \int_{t-s}^t |K(\tau+s,s)|x^2(\tau)d\tau$$

Under conditions (3.11) the functional $V = V_1 + V_2$ satisfies the conditions of the theorem on asymptotic stability.

2 For certain Volterra-type equations with distributed delay the transformed equation can be obtained by differentiating both sides of the original equation. For example, we consider the scalar equation

$$\dot{x}(t) = -ax(t) - \int_0^t K(t-s)x(s)ds, \quad t \geq 0 \tag{3.12}$$

with initial condition $x(0) = x_0$. Here a is a given constant and $K(s)$ is a continuously differentiable function.

Equation (3.12) is a special case of (3.10). Therefore the stability conditions for this equation follow from (3.11) and have the form

$$a > \int_0^\infty |K(s)|ds, \quad \int_0^\infty s|K(s)|ds < \infty$$

However, these stability conditions, which require the kernel $K(s)$ to be absolutely integrable in $[0, \infty)$ and the first moment to be finite, may turn out to be too restrictive for some equations. We shall therefore establish stability conditions using another representation for the transformed equations.

Differentiating both sides of (3.12) with respect to t, we obtain the following transformed system of two equations for $x_1 = x, x_2 = \dot{x}$, which is equivalent to the original one

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -ax_2 - bx_1 - \int_0^t \dot{K}(t-s)x(s)ds,\ b = K(0) \tag{3.13}$$

This means that the auxiliary system has the form

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -ay_2 - by_1$$

We take v for this system to be

$$v = 2by_1^2 + y_2^2 + (y_1 + y_2)^2 = v(y_1, y_2)$$

Consequently, $V_1(x_1, x_2) = v(y_1, y_2)$. From this and (3.13) it follows that $V = V_1 + V_2$, where

$$V_2 = (a+1)\int_0^\infty |\dot{K}(s)| \int_{t-s}^t x^2(s_1)ds_1$$

Then, by (3.13), $\dot{V}$ satisfies the estimate

$$\frac{1}{2}\dot{V} \leqslant \gamma_5 x^2 + \gamma_6 y^2,\ \ \gamma_5 = -aK(0) + (1+a)\int_0^\infty |\dot{K}(s)|ds,\ \ \gamma_6 = -a + \int_0^\infty |\dot{K}(s)|ds$$

It follows that the asymptotic stability conditions have the form $\gamma_5 < 0, \gamma_6 < 0$. The two conditions will be satisfied if the rate of change of $K(s)$ is small enough. In particular, if $K(s)$ is constant, then the sufficient stability conditions obtained for (3.13) become necessary and sufficient.

4. DISSIPATIVE SYSTEMS

Consider the system of equations

$$\dot{x}(t) = F(t, x(t-h)),\quad t \geqslant 0,\quad x \in R^n,\quad h \geqslant 0 \tag{4.1}$$

Here F: $[0, \infty) \times R^n \to R^n$ is a continuous function differentiable with respect to the second argument and such that $F(t, 0) \equiv 0$ and the Lipschitz condition

$$\|F(t, x_1) - F(t, x_2)\| \leqslant L\|x_1 - x_2\|,\quad x_i \in R^n \tag{4.2}$$

is satisfied, $\|\cdot\|$ being a norm in R^n. We also assume that the partial derivative matrix $f,(t,x) = \partial F(t,x)/\partial x$ is strictly dissipative uniformly in $t \geqslant 0, x \in D \subset R^n$ (where D is a neighbourhood of the origin). The latter means that $\|\exp(\tau f(t,x))\|_1 \leqslant \exp(-a\tau)$ for all $\tau \geqslant 0, t \geqslant 0, x \in D$ for some $a > 0$. We denote by $\|\cdot\|_1$ the operator norm generated by the vector norm $\|\cdot\|$ in R^n. We have $a < -\sup_{t,x}\gamma(f(t,x)), t \geqslant 0, x \in D$, where the logarithmic norm $\gamma(f)$ of f is defined by the equation

$$\gamma(f) = \lim_{\Delta\to+0} \frac{1}{\Delta}[\|I + \Delta f\|_1 - 1] \tag{4.3}$$

(I is the $(n \times n)$ identity matrix) [14, 15].

We emphasize that f should be dissipative with respect to some matrix norm $\|\cdot\|_1$ generated by any vector norm $\|\cdot\|$ in R^n. In particular, if $f(t,x)$ is negative definite uniformly in $t \geqslant 0$ and $x \in D$ (i.e. $y'f(t,y)y \leqslant -Cy'y$ for any $t \geqslant 0, x \in D\ y \in R^n$), then it is strictly dissipative with respect to the Euclidean norm and $\gamma(f) \leqslant -C$.

We shall establish stability conditions for the trivial solution of (4.1).

On adding and subtracting $F(t, x(t))$ on the right-hand side of (4.1), we conclude that the auxiliary system has the form

$$\dot{y}(t) = F(t, y(t))$$

and we take $v = \|y\|$ as the function v for this system. Then $V_1 = \|x(t)\|$. For any norm we have [14–16]

$$d^+ / dt\|x(t)\| = Q[x(t), \dot{x}(t)] \tag{4.4}$$

$$Q[x(t), \dot{x}(t)] = \lim_{\Delta\to+0} \tfrac{1}{\Delta}[\|x(t)+\Delta x(t)\| - \|x(t)\|]$$

where d^+/dt is the right-hand upper derivative. Replacing $\dot{x}$ in (4.4) by the right-hand side of (4.1), we obtain

$$\frac{d^+}{dt} V_1 \leq \lim_{\Delta\to+0} \frac{1}{\Delta}[\|x(t)+\Delta F(t,x(t))\| - \|x(t)\|] + \|F(t,x(t)) - F(t,x(t-h))\| \tag{4.5}$$

The limit in (4.5), which is equal to $Q[x(t), F(t, x(t))]$, satisfies the estimate

$$Q|x(t), \quad F(t,x(t))| \leq \gamma(f)\|x(t) \leq -a\|x(t)\| \tag{4.6}$$

Indeed, using (4.4), one can interpret $Q[x(t), F(t, x(t))]$ as the right-hand upper derivative $d^+\|y\|/dt$ of $\|y\|$ because of the trajectories of the system at the point $y = x(t)$, i.e.

$$\frac{d^+\|y\|}{dt} = Q[y, F(t,y)] \tag{4.7}$$

Furthermore,

$$F(t,y) = \int_0^1 f(t,sy)dsy \tag{4.8}$$

Besides, by [14–18]

$$\gamma(f) = \sup_y \|y\|^{-1} Q(y, fy), \quad \|y\| \neq 0, \quad y \in R^n \tag{4.9}$$

The estimate (4.6) follows from (4.7)–(4.9) and the convexity of the logarithmic norm, since

$$\frac{d^+\|y\|}{dt} = Q\left[y, \int_0^t f(t,sy)dsy\right] \leq \gamma\left[\int_0^1 f(t,sy)ds\right]\|y\| \leq \int_0^t \gamma[f(t,sy)]ds\|y\| \leq -a\|y\|$$

Inequalities (4.2), (4.5), and (4.6) mean that

$$\frac{d^+}{dt} V_1 \leq -a\|x(t)\| + L\|x(t) - x(t-h)\| = -a\|x(t)\| + L\left\|\int_{t-h}^t F(s, x(s-h))ds\right\| \leq$$

$$\leq L^2 \int_{t-2h}^{t-h} \|x(s)\|ds - a\|x(t)\|$$

It follows that

$$V_2 = L^2\left(\int_{t-2h}^{t-h} ds \int_s^{t-h} \|x(s_1)\|ds_1 + h\int_{t-h}^t \|x(s)\|ds\right)$$

and $d^+V/dt \leq -(a - hL^2)\|x(t)\|$, where $V = V_1 + V_2$. It has therefore been established that, under the assumptions made above, the trivial solution of (4.1) is uniformly asymptotically stable if $a > hL^2$.

For scalar equations of the form (4.1) this stability condition actually follows from [1], where it was established by the Lyapunov-function method for $n = 1$.

We also remark that the stability conditions for (1.1) established in Section 1 can be obtained using the argument of the present section, which can also be used when studying the stability of non-linear systems with arbitrary delay.

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